

NONNOETHERIAN GEOMETRY AND TORIC SUPERPOTENTIAL ALGEBRAS

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ABSTRACT. We show that nonnoetherian subalgebras of affine coordinate rings can be realized geometrically as affine varieties that contain positive dimensional subvarieties which are identified as closed points. We introduce the notion of the ‘geometric dimension’ of a point, and characterize the unique largest subset for which the closed points are zero dimensional.

The following application is then considered: Let A be a non-cancellative superpotential algebra of a brane tiling quiver Q , and suppose a cancellative algebra A' (a ‘superconformal quiver theory’) is obtained by contracting (‘Higgsing’) an adequate set of arrows in Q to vertices. We show that under a certain new isomorphism, the nonnoetherian center Z of A will be generated by the intersection of the cycles in Q , and birational to the noetherian ring generated by the union of these cycles (the ‘mesonic chiral ring’). Further, we show that the latter ring will be isomorphic to the center of A' , and therefore Z will be an affine toric Gorenstein singularity with a positive dimensional closed point.

1. INTRODUCTION

The purpose of this paper is two-fold: first, to introduce a new framework for understanding the geometry of infinitely-generated subalgebras of affine coordinate rings in some non-abstract sense; and second, to apply this framework to geometries that arise from certain superpotential quiver algebras.

A superpotential algebra of a cancellative brane tiling (also called a dimer model) is a type of quiver algebra with potential that has the cancellation property (Definition 3.6). It is now well known that these algebras are noncommutative crepant resolutions and 3-Calabi-Yau algebras with 3-dimensional normal toric Gorenstein centers [MR, D, Br, M, Bo, B]. Much less is understood, however, about superpotential algebras of non-cancellative brane tilings. In contrast to cancellative brane tilings, they are not finitely-generated modules over their centers, and their centers are nonnoetherian. In

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this paper we study the central geometry of these non-cancellative algebras, as it is of interest from both a mathematical and a string theory perspective.¹

We briefly outline our main results. The results in section 1 are about nonnoetherian subalgebras of affine coordinate rings, and may be of independent interest from superpotential algebras.

Let R be a subalgebra of an affine coordinate ring S . We introduce the subsets

$$U := \{\mathfrak{n} \in \text{Max } S \mid R_{\mathfrak{n} \cap R} = S_{\mathfrak{n}}\} \quad \text{and} \quad W := \left\{ \mathfrak{n} \in \text{Max } S \mid \sqrt{(\mathfrak{n} \cap R)S} = \mathfrak{n} \right\},$$

and show the following.

Theorem A. *Suppose R is a subalgebra of an affine k -algebra S . Then the map*

$$\phi : \text{Max } S \rightarrow \text{Max } R, \quad \mathfrak{q} \mapsto \mathfrak{q} \cap R,$$

is injective on U , and W is the unique largest subset of $\text{Max } S$ that ϕ is injective on. In particular, $U \subseteq W$. If $U \neq \emptyset$ then $\text{Max } S$ and $\text{Max } R$ are isomorphic on open dense subsets, and thus birationally equivalent.

Proposition B. *Let R' be a subalgebra of S , I an ideal of S , and form the algebra*

$$R = k[R', I].$$

Then U (hence W) contains the open subset $Z(I)^c$ of $\text{Max } S$. Furthermore, if $I \subset S$ is a non-maximal ideal and

$$R = k[I] = k + I,$$

then $W = U = Z(I)^c$, and the set $V(I)$ is a closed point in $\text{Spec } R$.

We then introduce the notions of depiction and geometric dimension: we say R is depicted by S if $U \neq \emptyset$ and the map $\phi : \text{Max } S \rightarrow \text{Max } R$, $\mathfrak{n} \mapsto \mathfrak{n} \cap R$, is surjective. Furthermore, for $\mathfrak{p} \in \text{Spec } R$ let $\text{codim}_S \mathfrak{p}$ denote the length of a longest chain of distinct prime ideals of R contained in \mathfrak{p} that lifts to a chain of prime ideals of S , and set $\dim_S \mathfrak{p} := \dim_S R - \text{codim}_S \mathfrak{p}$ (Definition 2.11). The geometric dimension of \mathfrak{p} is then defined to be the supremum

$$\dim^\circ \mathfrak{p} = \sup \{ \dim_S \mathfrak{p} \mid S \text{ a depiction of } R \}.$$

The following theorem relates $\dim_S \mathfrak{p}$ to the set U , and implies that the geometric dimension of a point will always be finite and bounded by the transcendence degree of $\text{Frac } R$ over k .

Theorem C. *Suppose R is depicted by S . Then*

$$\dim_S R = \text{trdeg}_k \text{Frac } R = \dim S.$$

¹In 4-dimensional theories these algebras often correspond to non-superconformal quiver gauge theories, which are perfectly reasonable physical theories. In 3-dimensional Chern-Simons quiver gauge theories, cancellative and non-cancellative brane tilings are usually on an equal footing; see for example [MS, BR].

If $\mathfrak{p} \in \operatorname{Spec} R$ and $\mathfrak{q} \in \phi^{-1}(\mathfrak{p})$, then

$$\dim \mathfrak{q} \leq \dim_S \mathfrak{p} \leq \dim S,$$

with equality on the left if $Z(\mathfrak{q}) \cap U \neq \emptyset$.

In section 2 we turn our attention to quiver algebras:

Theorem D. Suppose $A = kQ/I$ is a finitely-generated quiver algebra that admits an impression $\tau : A \hookrightarrow M_{|Q_0|}(B)$ (Definition 3.2), with B an affine integral domain and $\tau(e_i) = E_{ii}$ for each $i \in Q_0$. For $p \in e_j A e_i$, define $\bar{\tau}(p)$ by $\tau(p) = \bar{\tau}(p) E_{ji}$. Then the center of A is isomorphic to

$$(1) \quad R := \{r \in B \mid r 1_d \in \operatorname{im} \tau\} = k \left[\bigcap_{i \in Q_0} \bar{\tau}(e_i A e_i) \right] \subseteq B$$

and is depicted by

$$(2) \quad S := k \left[\bigcup_{i \in Q_0} \bar{\tau}(e_i A e_i) \right] \subseteq B.$$

We then characterize the central geometry of a class of superpotential algebras of non-cancellative brane tilings. To do this, we define a k -homomorphism that turns a set of arrows into vertices, called a contraction (Definitions 3.9 and 3.11). This formalizes an operation known as ‘Higgsing’ in quiver gauge theories (see Remark 3.10).

Theorem E. Let $\psi : A \rightarrow A'$ be an adequate contraction between superpotential algebras of brane tilings, where A' is cancellative and A is not. Further suppose A' admits an impression (τ', B) , with B a polynomial ring and $\bar{\tau}'(a) \in B$ a monomial for each $a \in Q'_1$. Define the k -homomorphism $\tau : A \rightarrow M_{|Q_0|}(B)$ by

$$\bar{\tau}(a) := \bar{\tau}'(\psi(a)) \text{ for each } a \in e_j A e_i, \ i, j \in Q_0,$$

and let R, S and R', S' be as in (1), (2) with τ and τ' respectively. Then

$$Z \cong R \subsetneq S = S' = R' \cong Z'.$$

Furthermore, R is depicted by S . In particular, the ‘mesonic chiral ring’ of A , namely S , is a depiction of its center Z .

As a corollary, we conclude that the nonnoetherian center Z of A is birational to the normal toric Gorenstein singularity $S = S'$ and contains a positive dimensional subvariety that is identified as a single (closed) point.

Finally, in Proposition 3.20 we give an infinite family of non-cancellative brane tilings for which Theorem E applies.

Notation: Throughout R is a subalgebra of an affine integral domain S , both of which contain an algebraically closed field k . We will denote by $\dim R$ the Krull dimension of R ; by $\operatorname{Frac} R$ the ring of fractions of R ; by $\operatorname{Max} R$ the set of maximal

ideals of R ; and by $\text{Spec } R$ either the set of prime ideals of R or the affine k -scheme with global sections R . For $\mathfrak{a} \subset R$ we denote by $Z(\mathfrak{a}) := \{\mathfrak{m} \in \text{Max } R \mid \mathfrak{m} \supseteq \mathfrak{a}\}$, $V(\mathfrak{a}) := \{\mathfrak{p} \in \text{Spec } R \mid \mathfrak{p} \supseteq \mathfrak{a}\}$, the respective Zariski-closed sets of $\text{Max } R$ and $\text{Spec } R$. Finally, Q_ℓ denotes the paths of length ℓ in a quiver Q .

2. NONNOETHERIAN GEOMETRY: POSITIVE DIMENSIONAL AND NONLOCAL POINTS

2.1. The largest subset. We begin with the following basic fact.

Lemma 2.1. *If $\mathfrak{q} \in \text{Max } S$ then $\mathfrak{q} \cap R \in \text{Max } R \subset \text{Spec } R$.*

Proof. Since S is finitely-generated over an algebraically closed field, $S/\mathfrak{q} \cong k$, and thus since $1_S \in R$, the composition $\psi : R \hookrightarrow S \rightarrow S/\mathfrak{q}$ is an epimorphism. Therefore $R/\ker \psi \cong k$, and so $\mathfrak{q} \cap R = \ker \psi \in \text{Max } R$. \square

The embedding $\iota : R \hookrightarrow S$ induces the morphism of schemes

$$(\phi, \phi^b) : (\text{Spec } S, \mathcal{O}_{\text{Spec } S}) \longrightarrow (\text{Spec } R, \mathcal{O}_{\text{Spec } R}),$$

where $\phi : \text{Spec } S \rightarrow \text{Spec } R$ is given by $\mathfrak{q} \mapsto \mathfrak{q} \cap R$.² We introduce the following subsets of the variety $\text{Max } S$.

Definition 2.2. For $\mathfrak{n} \in \text{Max } S$, set $\mathfrak{m} := \mathfrak{n} \cap R \in \text{Max } R$. Define the subsets

$$U_S := \{\mathfrak{n} \in \text{Max } S \mid R_{\mathfrak{m}} = S_{\mathfrak{n}}\}, \quad W_S := \left\{ \mathfrak{n} \in \text{Max } S \mid \sqrt{\mathfrak{m}S} = \mathfrak{n} \right\}.$$

We will omit the subscript S when S is fixed. Recall that S is an overring of a domain R if $R \subseteq S \subseteq \text{Frac } R$.

Lemma 2.3. *If U is nonempty then S is an overring of R . In particular, the function field of $\text{Spec } R$ equals the function field of $\text{Spec } S$. Furthermore, U contains a nonempty open subset of $\text{Max } S$.*

Proof. Suppose $\mathfrak{n} \in U$. Then since S is an integral domain, $S \subseteq \text{Frac } S = \text{Frac}(S_{\mathfrak{n}}) = \text{Frac}(R_{\mathfrak{n} \cap R}) = \text{Frac } R$.

We now show that U contains a nonempty open subset of $\text{Max } S$. We first claim that if A is a subalgebra of B , $\mathfrak{n} \in \text{Max } A$, and $\mathfrak{n}B \cap A \neq A$, then $A_{\mathfrak{n}} \subseteq B_{\mathfrak{n}B}$. Consider $\frac{a}{b} \in A_{\mathfrak{n}}$ with $a, b \in A \subseteq B$ and $b \notin \mathfrak{n}$. Then $b \notin \mathfrak{n}B$: suppose $b \in \mathfrak{n}B$. Then $b \in \mathfrak{n}B \cap A$. But $\mathfrak{n}B \cap A \supseteq \mathfrak{n} \in \text{Max } A$, and $\mathfrak{n}B \cap A \neq A$ by assumption, so $\mathfrak{n}B \cap A = \mathfrak{n}$, whence $b \in \mathfrak{n}$, contrary to our assumption. Therefore $b \notin \mathfrak{n}B$, so $\frac{a}{b} \in B_{\mathfrak{n}B}$, proving our claim.

Now suppose $\{a_i\}_{i \in I}$ is a generating set for S and set $J := \{j \in I \mid a_j \notin R\}$. Since $S \subset \text{Frac } R$ by Lemma 2.3, for each $j \in J$ there is a $c_j \in R$ such that $a_j c_j \in R$. The subset

$$U' := \{\mathfrak{n} \in \text{Max } S \mid c_j \notin \mathfrak{n} \ \forall j \in J\}$$

²This follows since $R_{\mathfrak{q} \cap R} \subseteq S_{\mathfrak{q}}$: if $a \in R_{\mathfrak{q} \cap R}$ then $a = \frac{b}{c}$ with $b, c \in R \subset S$, $c \notin \mathfrak{q} \cap R$, so $c \notin \mathfrak{q}$, whence $a \in S_{\mathfrak{q}}$.

of $\text{Max } S$ is nonempty and open since $|J| \leq |I| < \infty$. Suppose $\mathfrak{n} \in U'$ and $\mathfrak{m} = \mathfrak{n} \cap R$. Then $c_j \notin \mathfrak{n}$, hence $c_j \notin \mathfrak{m}$, for each $j \in J$. Therefore $S \subseteq R_{\mathfrak{m}}$. But then $S \subseteq R_{\mathfrak{m}} \subseteq S_{\mathfrak{n}}$, so by our claim above we have

$$S_{\mathfrak{n}} \subseteq (R_{\mathfrak{m}})_{\mathfrak{n}R_{\mathfrak{m}}} \subseteq (S_{\mathfrak{n}})_{\mathfrak{n}S_{\mathfrak{n}}} = S_{\mathfrak{n}},$$

which yields $S_{\mathfrak{n}} = (R_{\mathfrak{m}})_{\mathfrak{n}R_{\mathfrak{m}}} = R_{\mathfrak{m}}$. Therefore $U' \subseteq U$. \square

In the following theorem we show that W is similar in spirit to the Azumaya locus of A when A is a noncommutative algebra, module-finite over its center Z . Recall that if $\mathfrak{n}, \mathfrak{n}' \in \text{Max } A$ and $\mathfrak{n} \cap Z = \mathfrak{n}' \cap Z$ is in the Azumaya locus of A , then $\mathfrak{n} = \mathfrak{n}'$.³ For a subset Y of $\text{Max } S$, set $Y^c := \text{Max } S \setminus Y$.

Theorem 2.4. *The map*

$$\phi : \text{Max } S \rightarrow \text{Max } R, \quad \mathfrak{q} \mapsto \mathfrak{q} \cap R,$$

is injective on U , and W is the unique largest subset of $\text{Max } S$ that ϕ is injective on. In particular, $U \subseteq W$. If $U \neq \emptyset$ then $\text{Max } S$ and $\text{Max } R$ are isomorphic on open dense subsets, and thus birationally equivalent.

Proof. We first show that ϕ is injective on U : if $\mathfrak{n}, \mathfrak{n}' \in U$ and $\mathfrak{n} \cap R = \mathfrak{n}' \cap R$, then $S_{\mathfrak{n}} = R_{\mathfrak{n} \cap R} = R_{\mathfrak{n}' \cap R} = S_{\mathfrak{n}'}$, so $S_{\mathfrak{n}}$ has unique maximal ideal $\mathfrak{n} = \mathfrak{n}'$.

We now claim that $\mathfrak{n} \in W^c$ if and only if there is a point $\mathfrak{n}' \in \text{Max } S$, not equal to \mathfrak{n} , such that $\phi(\mathfrak{n}) = \phi(\mathfrak{n}')$. First note that for $\mathfrak{m} \in \text{Max } R$, $\mathfrak{m} \subseteq \mathfrak{m}S \subseteq \sqrt{\mathfrak{m}S} \subseteq \mathfrak{n}$, so $\mathfrak{m} \subseteq \sqrt{\mathfrak{m}S} \cap R \subseteq \mathfrak{n} \cap R = \mathfrak{m}$, which yields

$$(3) \quad \sqrt{\mathfrak{m}S} \cap R = \mathfrak{m}.$$

Set $\mathfrak{m} := \mathfrak{n} \cap R$ and suppose $\mathfrak{n} \neq \sqrt{\mathfrak{m}S}$. Since S is Jacobson, $\sqrt{\mathfrak{m}S} = \bigcap_{\sqrt{\mathfrak{m}S} \subseteq \mathfrak{q} \in \text{Max } S} \mathfrak{q}$, so there exists a maximal ideal $\mathfrak{n}' \neq \mathfrak{n}$ of S such that $\sqrt{\mathfrak{m}S} \subseteq \mathfrak{n}'$. But then by Lemma 2.1,

$$\mathfrak{m} \stackrel{(3)}{=} \sqrt{\mathfrak{m}S} \cap R \subseteq \mathfrak{n}' \cap R \in \text{Max } R,$$

so $\phi(\mathfrak{n}') = \mathfrak{n}' \cap R = \mathfrak{m} = \mathfrak{n} \cap R = \phi(\mathfrak{n})$.

Conversely suppose there are distinct points $\mathfrak{n}, \mathfrak{n}' \in \text{Max } S$ such that $\phi(\mathfrak{n}) = \phi(\mathfrak{n}')$. Then $\mathfrak{n} \cap R = \mathfrak{n}' \cap R =: \mathfrak{m}$, and so $\sqrt{\mathfrak{m}S} \subseteq \mathfrak{n} \cap \mathfrak{n}' \subsetneq \mathfrak{n}$.

Finally, $\text{Max } S$ is irreducible so U contains an open dense subset by Lemma 2.3. Therefore $\text{Max } S$ and $\text{Max } R$ are birationally equivalent since ϕ is injective on U . \square

Example 2.5. Let $S = k[x, y]$ and $R = k[x, xy, xy^2, \dots] = k + (x)S$. For any $b \in k$, the ideals $(x, y - b), (x) \in \text{Spec } S$ satisfy

$$(x, y - b) \cap R = (x) \cap R = (x, xy, xy^2, \dots) \in \text{Max } R,$$

so $(x, y - b) \in W^c$ by Theorem 2.4.

³If S is finitely-generated over k but R is not, then S will not be a finitely-generated R -module; this follows, for example, from the Artin-Tate lemma.

The following proposition generalizes the fact that for $\mathfrak{n} \in \text{Max } S$, $S = k + \mathfrak{n}$.

Proposition 2.6. *Let R' be a subalgebra of S , I an ideal of S , and form the algebra*

$$R = k[R', I].$$

Then U (hence W) contains the open subset $Z(I)^c$ of $\text{Max } S$. Furthermore, if $I \subset S$ is a non-maximal ideal and

$$R = k[I] = k + I,$$

then $W = U = Z(I)^c$, and the set $V(I)$ is a closed point in $\text{Spec } R$.

Proof. We claim that if $\mathfrak{q} \in \text{Spec } S$ does not contain I then $R_{\mathfrak{q} \cap R} = S_{\mathfrak{q}}$; in particular, if $\mathfrak{q} \in \text{Max } S$ then $\mathfrak{q} \in U$. Set $\mathfrak{p} := \mathfrak{q} \cap R$. Then $R_{\mathfrak{p}} \subseteq S_{\mathfrak{q}}$, so suppose $a \in S_{\mathfrak{q}}$, i.e., there is some $f, g \in S$, $g \notin \mathfrak{q}$, such that $a = \frac{f}{g}$. Since \mathfrak{q} does not contain I there is some $c \in I \setminus \mathfrak{q}$. Since $c, g \in S \setminus \mathfrak{q}$ and \mathfrak{q} is prime, we have $cg \in S \setminus \mathfrak{q}$. Since $c \in I$, $cg \in I \subset R$, so $cg \in R \setminus \mathfrak{p}$. But also $b := agc \in I \subset R$, and thus $a = \frac{b}{cg} \in R_{\mathfrak{p}}$.

For the second statement, clearly $I \in \text{Max } R$. Suppose I is not a maximal ideal of S . Let $\mathfrak{n} \in Z(I)$. Then $\mathfrak{n} \supseteq I$, so $\mathfrak{n} \cap R \supseteq I \cap R = I \in \text{Max } R$, so $\mathfrak{n} \cap R = I$. But then $\sqrt{I}S = IS = I \subsetneq \mathfrak{n}$, and so $Z(I) \subseteq W^c$. The converse follows from the previous paragraph, and so $W^c = Z(I)$. Since $Z(I) \supseteq U^c \supseteq W^c = Z(I)$, we also have $U^c = Z(I)$. \square

Note that U may properly contain $Z(I)^c$; for example, take $R' = S$.

Example 2.7. *A geometric picture.*

(i) Again let $S = k[x, y]$ and $R = k + (x)S$. By Proposition 2.6, we can form the space $\text{Max } R$ by declaring the line $\{x = 0\} = Z(x) \subset \mathbb{A}_k^2 = \text{Max } S$ to be a single (closed) point, while all other points, $U = \{x \neq 0\}$, remain unaltered.

(ii) Let $S = k[x, y, z]$ and $R = k[x, y, yz, yz^2, \dots] = k[x, (y)S]$. We can form the space $\text{Max } R$ by declaring each line $\{x = c, y = 0\} = Z(x - c, y) \subset \mathbb{A}_k^3 = \text{Max } S$ to be a point, while all other points, $U = \{y \neq 0\}$, remain unaltered.

The following definition formalizes these ‘geometric pictures’.

Definition 2.8. We say S is a *depiction* of R (or R is depicted by S) if $U \neq \emptyset$ and the map $\phi : \text{Max } S \rightarrow \text{Max } R$, $\mathfrak{n} \mapsto \mathfrak{n} \cap R$, is surjective.

Question 2.9. If R admits a depiction, then does R admit a unique maximal depiction with respect to inclusion?

A partial answer to this question is given in the next proposition. We say two elements $a, b \in S$ are coprime if their only common divisors are in k .

Proposition 2.10. *Suppose S is a depiction of R with the property that $a, b \in S$ are not coprime whenever $a|b^n$ for some $n \geq 1$. Then S is the unique maximal depiction of R .*

Proof. Suppose R admits depictions S and S' , S' has the ‘coprime property’, and assume to the contrary that $a \in S \setminus S'$. Since $U_S \subset \text{Max } S$ is nonempty, $\text{Frac } S = \text{Frac } R$ by Lemma 2.3. Therefore $a = \frac{b}{c}$ for some $b, c \in R$, which we can assume to be coprime in S' since $a \notin S'$. If c were a unit of R then $a = c^{-1}b \in R \subset S'$, a contradiction, so c is contained in at least one maximal ideal $\mathfrak{m} \in \text{Max } R$.

Suppose $c \in \mathfrak{p} \in \text{Max } R$. Since S is a depiction of R there is some $\mathfrak{q} \in \text{Max } S$ such that $\mathfrak{q} \cap R = \mathfrak{p}$. It follows that $c \in \mathfrak{q}$, so $b = ac \in \mathfrak{q}$, so $b \in \mathfrak{q} \cap R = \mathfrak{p}$. I.e., $c \in \mathfrak{p} \in \text{Max } R$ implies $b \in \mathfrak{p}$. Therefore $c \in \mathfrak{n} \in \text{Max } S'$ implies $c \in \mathfrak{n} \cap R \in \text{Max } R$, which implies $b \in \mathfrak{n} \cap R \subseteq \mathfrak{n}$. Since S' is a depiction, there is some $\mathfrak{n} \in \text{Max } S'$ such that $\mathfrak{n} \cap R = \mathfrak{m} \ni c$, so the intersection $\bigcap_{c \in \mathfrak{n} \in \text{Max } S'} \mathfrak{n}$ is nonempty, and thus contains b . Consequently

$$b \in \bigcap_{c \in \mathfrak{n} \in \text{Max } S'} \mathfrak{n} \stackrel{(i)}{=} \sqrt{(c)S'},$$

where (i) holds since S' is Jacobson. Therefore $c|b^n$ for some $n \geq 1$. But b and c were chosen to be coprime, contradicting our assumption that S' has the coprime property. Therefore $S \subseteq S'$. \square

2.2. Geometric dimensions of points. *Throughout this section we assume that R is depicted by S .* We introduce the following modifications of height and Krull dimension.

Definition 2.11. Let $\mathfrak{p} \in \text{Spec } R$. Denote by $\text{codim}_S \mathfrak{p}$ the length d of a longest chain of prime ideals of R , $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_d = \mathfrak{p}$, that lifts to a chain of prime ideals of S , $\mathfrak{q}_0 \subseteq \cdots \subseteq \mathfrak{q}_d$, in the sense that $\mathfrak{p}_i = \mathfrak{q}_i \cap R$. Set

$$\dim_S R := \sup \{ \text{codim}_S \mathfrak{p} \mid \mathfrak{p} \in \text{Spec } R \}, \quad \dim_S \mathfrak{p} := \dim_S R - \text{codim}_S \mathfrak{p}.$$

We then define the *geometric dimension* of \mathfrak{p} to be

$$\dim^\circ \mathfrak{p} = \sup \{ \dim_S \mathfrak{p} \mid S \text{ a depiction of } R \}.$$

Our main result of this section is Theorem 2.19. We first give a couple examples.

Example 2.12. *Geometric dimension.* Again consider $S = k[x, y]$, $R = k + (x)S$. S is the unique largest depiction of R by Proposition 2.10. The ideal $(x)S \cap R$ of R has height at least 2 since $0 \subsetneq (y)S \cap R \subsetneq (x)S \cap R$ is a chain of prime ideals in R . Fortunately this chain does not lift to a chain of prime ideals of S : we have $(y)S \cap R = (xy)S \cap R$, but $(y)S \not\subseteq (x)S$, and $(xy)S$ is not prime in S . Therefore the geometric dimension of the closed point $(x)S \cap R \in \text{Max } R$ is 1, noting that it lifts to the line $\{x = 0\} = Z(x) \subset \mathbb{A}_k^2$.

Example 2.13. *A zero-dimensional nonlocal point.* Let $S = k[X]$ be the coordinate ring for an algebraic variety X , and let $\mathfrak{n}_1, \dots, \mathfrak{n}_r$ be maximal ideals of S . Then by Proposition 2.6,

$$R = k + \prod_{i=1}^r \mathfrak{n}_i$$

is the coordinate ring for a space which is identical to X , with the exception that the r points $\mathbf{n}_1, \dots, \mathbf{n}_r$ are identified as one single point. S is a depiction of R , and U^c is zero dimensional.

Lemma 2.14. *If $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2$, $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$, and $\mathfrak{p}_i = \mathfrak{q}_i \cap R$, then $\mathfrak{q}_1 \subsetneq \mathfrak{q}_2$.*

Proof. If $a \in \mathfrak{p}_2 \setminus \mathfrak{p}_1$ then $a \in R$, so $a \notin \mathfrak{q}_1$. But $\mathfrak{p}_2 \subset \mathfrak{q}_2$, so $a \in \mathfrak{q}_2 \setminus \mathfrak{q}_1$. \square

Lemma 2.15. *Let $\mathfrak{p} \in \text{Spec } R$, and $\mathfrak{q} \in \phi^{-1}(\mathfrak{p})$. Then $Z(\mathfrak{p}) \cap \phi(U) \neq \emptyset$ if and only if $Z(\mathfrak{q}) \cap U \neq \emptyset$.*

Proof. If $\mathbf{n} \in U$ contains \mathfrak{q} then $\mathfrak{p} = \mathfrak{q} \cap R \subseteq \mathbf{n} \cap R \in \phi(U)$. Conversely, suppose $\mathbf{m} \in Z(\mathfrak{p}) \cap \phi(U)$. Since S is a depiction of R and $\mathbf{m} \in Z(\mathfrak{p})$, there is some $\mathbf{n} \in Z(\mathfrak{q})$ such that $\phi(\mathbf{n}) = \mathbf{m}$. Furthermore, since $\mathbf{m} \in \phi(U)$ there is some $\mathbf{n}' \in U$ such that $\phi(\mathbf{n}') = \mathbf{m}$. But ϕ is injective on U by Theorem 2.4, and so $\mathbf{n} = \mathbf{n}'$. Therefore $\mathbf{n} \in Z(\mathfrak{q}) \cap U$. \square

For the following, set $\mathfrak{p}_m := \mathfrak{p}R_m$.

Lemma 2.16. *Let $\mathfrak{p} \in \text{Spec } R$. Consider a maximal chain of distinct prime ideals in R containing \mathfrak{p} ,*

$$\mathfrak{p}_0 \subsetneq \dots \subsetneq \mathfrak{p}_d = \mathfrak{p}.$$

Suppose $\mathbf{m} \in Z(\mathfrak{p}) \cap \phi(U)$ and let $\mathbf{n} \in \phi^{-1}(\mathbf{m})$. Then $\mathfrak{p}S_n \in \text{Spec } S_n$, and

$$(4) \quad \mathfrak{p}_0S_n \subsetneq \dots \subsetneq \mathfrak{p}_dS_n$$

is a maximal chain of prime ideals in S_n containing $\mathfrak{p}S_n$ with the property that

$$(5) \quad \mathfrak{p}_iS_n \cap R = \mathfrak{p}_i \quad \text{for } 0 \leq i \leq d.$$

Proof. (i) Let $\mathfrak{p} \subsetneq \mathfrak{p}'$ be prime ideals in R . We claim that $\mathfrak{p}_m \subsetneq \mathfrak{p}'_m$, and so it suffices to show that $\mathfrak{p}_m \neq \mathfrak{p}'_m$. Assume to the contrary that these ideals are equal and let $a \in \mathfrak{p}' \setminus \mathfrak{p}$. Then there exists a $d \in \mathfrak{p}$ and $\frac{b}{c} \in R_m$, with $b, c \in R$, $c \notin \mathfrak{m} \supset \mathfrak{p}$, such that $a = \frac{db}{c} \in \mathfrak{p}_m$. But then $ac = db \in \mathfrak{p}$ while a and c are in $R \setminus \mathfrak{p}$, contradicting the fact that \mathfrak{p} is a prime ideal in R .

(ii) To show that $\mathfrak{p}S_n \cap R = \mathfrak{p}$, consider

$$\mathfrak{p} \subseteq \mathfrak{p}S_n \cap R \stackrel{(i)}{=} \mathfrak{p}R_m \cap R = \mathfrak{p},$$

where (i) follows since $\mathbf{m} \in \phi(U)$.

(iii) Since $\mathbf{n} \in U$, $\mathfrak{p}_m = \mathfrak{p}S_n$, and so \mathfrak{p}_m is an ideal in S_n . To show that \mathfrak{p}_m is prime in S_n , consider $ab \in \mathfrak{p}_m$ and $a, b \in S_n = R_m$. Then $a = \frac{a_1}{a_2}, b = \frac{b_1}{b_2} \in R_m$, with $a_i, b_i \in R$ and $a_2, b_2 \notin \mathfrak{m}$, so

$$R \ni a_1b_1 = a_2b_2ab \in \mathfrak{p}_m.$$

Thus $a_1b_1 \in R \cap \mathfrak{p}_m = \mathfrak{p}$, so a_1 or b_1 is in \mathfrak{p} , and so a or b is in \mathfrak{p}_m . \square

Proposition 2.17. *Let $\mathfrak{p} \in \text{Spec } R$ be such that $Z(\mathfrak{p}) \cap \phi(U) \neq \emptyset$. Given any maximal chain of distinct prime ideals of R containing \mathfrak{p} ,*

$$\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_d = \mathfrak{p},$$

there exists a maximal chain of distinct prime ideals of S ,

$$\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_d$$

with the property that $\mathfrak{q}_i \cap R = \mathfrak{p}_i$ for each $0 \leq i \leq d$. Therefore $\text{codim}_S \mathfrak{p} \leq \text{codim } \mathfrak{p}$.

Proof. Consider $\mathfrak{n} \in \text{Max } S$ satisfying $\mathfrak{n} \cap R \in Z(\mathfrak{p}) \cap \phi(U)$. Set $\mathfrak{q}_i := \mathfrak{p}_i S_{\mathfrak{n}} \cap S$. Then \mathfrak{q}_i is an ideal in S :

$$\mathfrak{q}_i \subseteq \mathfrak{q}_i S = (\mathfrak{p}_i S_{\mathfrak{n}} \cap S) S \subseteq \mathfrak{p}_i S_{\mathfrak{n}} S \cap S = \mathfrak{p}_i S_{\mathfrak{n}} \cap S = \mathfrak{q}_i,$$

which yields $\mathfrak{q}_i = \mathfrak{q}_i S$. \mathfrak{q}_i is prime in S : suppose $ab \in \mathfrak{q}_i$ with $a, b \in S \subset S_{\mathfrak{n}}$. Then $ab \in \mathfrak{p}_i S_{\mathfrak{n}}$, so a or b is in $\mathfrak{p}_i S_{\mathfrak{n}}$ by Lemma 2.16, and so a or b is in $\mathfrak{p}_i S_{\mathfrak{n}} \cap S = \mathfrak{q}_i$.

Furthermore, $\mathfrak{q}_i \cap R = \mathfrak{p}_i S_{\mathfrak{n}} \cap R \stackrel{(5)}{=} \mathfrak{p}_i$. Since $\mathfrak{p}_i S_{\mathfrak{n}} \subsetneq \mathfrak{p}_{i+1} S_{\mathfrak{n}}$ by (4), we have $\mathfrak{q}_i = \mathfrak{p}_i S_{\mathfrak{n}} \cap S \subseteq \mathfrak{p}_{i+1} S_{\mathfrak{n}} \cap S = \mathfrak{q}_{i+1}$. We have thus shown that there is a chain of prime ideals

$$\mathfrak{q}_0 \subseteq \cdots \subseteq \mathfrak{q}_d$$

with the property that $\mathfrak{q}_i \cap R = \mathfrak{p}_i$ for each $0 \leq i \leq d$, and these inclusions are strict by Lemma 2.14. \square

Proposition 2.18. *Let $\mathfrak{p} \in \text{Spec } R$ and $\mathfrak{q} \in \phi^{-1}(\mathfrak{p})$. Then $\text{codim } \mathfrak{q} \leq \text{codim}_S \mathfrak{p}$, with equality if $Z(\mathfrak{q}) \cap U \neq \emptyset$.*

Proof. Set $d := \text{codim } \mathfrak{q}$.

(i) First suppose $Z(\mathfrak{q}) \cap U \neq \emptyset$, i.e., \mathfrak{q} is contained in some $\mathfrak{n} \in U$. Set $\mathfrak{m} = \mathfrak{n} \cap R$. We claim that $\text{codim}_S \mathfrak{p} = \text{codim } \mathfrak{q}$, so by Proposition 2.17 and Lemma 2.14 it suffices to show that $\text{codim}_S \mathfrak{p} \geq \text{codim } \mathfrak{q}$. Consider a maximal chain of distinct prime ideals $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_d = \mathfrak{q}$. Since $\mathfrak{q} \subseteq \mathfrak{n}$, there is a chain of distinct prime ideals $\mathfrak{q}_0 S_{\mathfrak{n}} \subsetneq \cdots \subsetneq \mathfrak{q}_d S_{\mathfrak{n}}$ in $S_{\mathfrak{n}}$. Since $\mathfrak{n} \in U$, this is a chain of distinct prime ideals $\mathfrak{q}_0 R_{\mathfrak{m}} \subsetneq \cdots \subsetneq \mathfrak{q}_d R_{\mathfrak{m}}$ in $R_{\mathfrak{m}}$. Set $\mathfrak{t}_i := \mathfrak{q}_i R_{\mathfrak{m}}$; since \mathfrak{t}_i is a proper ideal in $R_{\mathfrak{m}}$, we have $\mathfrak{t}_i \cap R \subseteq \mathfrak{m}$.

Suppose $a \in \mathfrak{t}_{i+1} \setminus \mathfrak{t}_i$. Then there is some $c \in R \setminus \mathfrak{m}$ such that $ac \in R$ (clearing denominators). If $ac \in \mathfrak{t}_i$ then $c \in \mathfrak{t}_i$ since \mathfrak{t}_i is prime. But then $c \in \mathfrak{t}_i \cap R \subseteq \mathfrak{m}$, a contradiction. Therefore $ac \in \mathfrak{t}_{i+1} \cap R \setminus \mathfrak{t}_i \cap R$, and so the chain of ideals

$$\mathfrak{t}_0 \cap R \subsetneq \cdots \subsetneq \mathfrak{t}_d \cap R$$

is strict. Furthermore, these ideals are prime: if $ab \in \mathfrak{t}_i \cap R$ and $a, b \in R$ then $ab \in \mathfrak{t}_i$, so a or b is in \mathfrak{t}_i , so a or b is in $\mathfrak{t}_i \cap R$. Finally, $\mathfrak{p} = \mathfrak{t}_d \cap R$ since

$$\mathfrak{p} \subseteq \mathfrak{q} R_{\mathfrak{m}} \cap R \stackrel{(a)}{=} \mathfrak{q} S_{\mathfrak{n}} \cap R = \mathfrak{q} S_{\mathfrak{n}} \cap S \cap R \stackrel{(b)}{=} \mathfrak{q} \cap R = \mathfrak{p},$$

where (a) holds since $\mathfrak{n} \in U$ and (b) holds since $\mathfrak{q} \subseteq \mathfrak{n}$. This proves our claim.

(ii) Now suppose $Z(\mathfrak{q}) \cap U = \emptyset$. U contains a nonempty open set U' by Lemma 2.3, which is dense since $\text{Max } S$ is irreducible. Thus there exists a chain of distinct

prime ideals of S , $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_{d-1} \subsetneq \mathfrak{q}_d = \mathfrak{q}$, such that $Z(\mathfrak{q}_{d-1}) \cap U' \neq \emptyset$. Then $Z(\mathfrak{q}_{d-1}) \cap U \neq \emptyset$, say $\mathfrak{n} \in Z(\mathfrak{q}_{d-1}) \cap U$. Therefore by (i), $\text{codim}_S \mathfrak{p} \geq d - 1$. If $\mathfrak{q}_{d-1} \cap R = \mathfrak{q}_d \cap R$ then $\mathfrak{p} = \mathfrak{q}_{d-1} \cap R \subseteq \mathfrak{n} \cap R \in \phi(U)$, whence $\mathfrak{n} \cap R \in Z(\mathfrak{p}) \cap \phi(U)$, contrary to assumption by Lemma 2.15. Otherwise $\mathfrak{q}_{d-1} \cap R \subsetneq \mathfrak{p}$, and so $\text{codim}_S \mathfrak{p} \geq \text{codim } \mathfrak{q}$. \square

Theorem 2.19. *Suppose R is depicted by S . Then*

$$(6) \quad \dim_S R = \text{trdeg}_k \text{Frac } R = \dim S.$$

If $\mathfrak{p} \in \text{Spec } R$ and $\mathfrak{q} \in \phi^{-1}(\mathfrak{p})$, then

$$(7) \quad \dim \mathfrak{q} \leq \dim_S \mathfrak{p} \leq \dim S,$$

with equality on the left if $Z(\mathfrak{q}) \cap U \neq \emptyset$.

Proof. We first prove (6). By Lemma 2.3 and our assumption that $\text{Max } S$ is an algebraic variety, $\dim S = \text{trdeg}_k \text{Frac } S = \text{trdeg}_k \text{Frac } R$. Since U is nonempty, there is some $\mathfrak{n} \in U$ such that

$$\dim_S R \leq \dim S \stackrel{(i)}{=} \text{codim } \mathfrak{n} \stackrel{\text{Prop 2.18}}{=} \text{codim}_S(\mathfrak{n} \cap R) \leq \dim_S R,$$

where (i) follows since $\text{Max } S \ni \mathfrak{n}$ is an (irreducible) algebraic variety.

We now prove (7). $\dim_S \mathfrak{p} \leq \dim S$ holds by (6). Moreover,

$$\dim_S \mathfrak{p} = \dim_S R - \text{codim}_S \mathfrak{p} \stackrel{(6)}{=} \dim S - \text{codim}_S \mathfrak{p} \stackrel{\text{Lemma 2.14}}{\geq} \dim S - \text{codim } \mathfrak{q} = \dim \mathfrak{q}.$$

If $Z(\mathfrak{q}) \cap U \neq \emptyset$ then $\dim_S \mathfrak{p} = \dim \mathfrak{q}$ by (6) and Proposition 2.18. \square

Corollary 2.20. *The geometric dimension of a point $\mathfrak{p} \in \text{Spec } R$ is always finite, and bounded by the transcendence degree of $\text{Frac } R$ over k .*

3. CENTRAL GEOMETRY OF NON-CANCELLATIVE TORIC SUPERPOTENTIAL ALGEBRAS

Throughout let $A = kQ/I$ be a finitely-generated quiver algebra and let B be an affine integral domain containing k .

3.1. Depictions from quiver algebras. Denote by $E_{ji} \in M_{|Q_0|}(B)$ the matrix whose (ji) -th entry is 1 and all other entries zero. Given an algebra homomorphism $\tau : A \rightarrow M_{|Q_0|}(B)$ satisfying $\tau(e_i) = E_{ii}$ for each $i \in Q_0$, denote by $\bar{\tau} : e_j A e_i \rightarrow B$ the k -homomorphism defined by $\tau(p) = \bar{\tau}(p) E_{ji}$ for each $p \in e_j A e_i$.

Proposition 3.1. *Let $\tau : A \rightarrow M_{|Q_0|}(B)$ be a k -homomorphism that is an algebra homomorphism on each $e_i A e_i$, $i \in Q_0$, with $\tau(e_i) = E_{ii}$. Suppose there is a cycle $b \in A$ which contains each vertex as a subpath and whose τ -image is nonzero. Further suppose the map $\text{Max } B \rightarrow \text{Max } R$, $\mathfrak{q} \mapsto \mathfrak{q} \cap R$, is surjective. Then the subalgebra*

$$(8) \quad R := \{r \in B \mid r 1_d \in \text{im } \tau\} = k \left[\bigcap_{i \in Q_0} \bar{\tau}(e_i A e_i) \right] \subseteq B$$

is depicted by

$$(9) \quad S := k \left[\bigcup_{i \in Q_0} \bar{\tau}(e_i A e_i) \right] \subseteq B.$$

Proof. A is finitely-generated, so $|Q_1| < \infty$, so S is finitely-generated. Furthermore, S is a domain since it is a subalgebra of the domain B .

U_S is nonempty: Fix $i \in Q_0$. Let $b_i \in e_i A e_i$ be a cycle that contains each vertex e_j as a subpath, and let $c_i \in e_i A e_i$ be an arbitrary cycle. For each $j \in Q_0$, denote by b_j and d_j the respective cycles obtained by cyclically permuting b_i and $d_i := b_i c_i$ so that their heads and tails are at j . Then $\bar{\tau}(b_j) = \bar{\tau}(b_i) =: \beta$ and $\bar{\tau}(d_j) = \bar{\tau}(d_i) = \bar{\tau}(c_i)\beta$, since τ is an algebra homomorphism on $e_i A e_i$. Therefore β and $\bar{\tau}(c_i)\beta$ are in R . Let \mathfrak{q} be a point in the nonempty open subset of $\text{Max } B$ defined by $\beta \neq 0$. By Lemma 2.1, $\mathfrak{n} := \mathfrak{q} \cap S$ and $\mathfrak{m} := \mathfrak{n} \cap R$ are maximal ideals of S and R respectively, and $\beta \in R$ is invertible in the localization $R_{\mathfrak{m}}$. Consequently

$$\bar{\tau}(c_i) = \bar{\tau}(c_i)\beta^n \cdot \beta^{-n} \in R_{\mathfrak{m}}.$$

Since c_i was arbitrary, $S_{\mathfrak{n}} \subseteq R_{\mathfrak{m}}$, whence $S_{\mathfrak{n}} = R_{\mathfrak{m}}$.

The map $\phi : \text{Max } S \rightarrow \text{Max } R$, $\mathfrak{n} \mapsto \mathfrak{n} \cap R$, is surjective: Let $\mathfrak{m} \in \text{Max } R$. By assumption there is some $\mathfrak{q} \in \text{Max } B$ such that $\mathfrak{q} \cap R = \mathfrak{m}$. But $\mathfrak{n} := \mathfrak{q} \cap S \in \text{Max } S$ and $\mathfrak{n} \cap R = (\mathfrak{q} \cap S) \cap R = \mathfrak{m}$. \square

We are interested in cases where the center of A is isomorphic to R . For the remainder of this section, let R and S be as in (8) and (9). The following definition was introduced in [B] to study a class of superpotential algebras of cancellative brane tilings (Definition 3.6).

Definition 3.2. An *impression* (τ, B) of A is a commutative affine k -algebra B and an algebra monomorphism $\tau : A \hookrightarrow M_d(B)$ such that (i) for each \mathfrak{q} in some open dense subset $U \subseteq \text{Max } B$, the composition with the evaluation map

$$A \xrightarrow{\tau} M_d(B) \xrightarrow{\epsilon_{\mathfrak{q}}} M_d(B/\mathfrak{q})$$

is surjective, and (ii) the morphism $\text{Max } B \rightarrow \text{Max } R$, $\mathfrak{q} \mapsto \mathfrak{q} \cap R$, is surjective [B, Definition and Lemma 2.1].

An impression is useful in part because it determines the center Z of A , and if A is a finitely-generated Z -module then it determines all simple A -module isoclasses of maximal k -dimension [B, Proposition 2.5].

Theorem 3.3. If $\tau : A \hookrightarrow M_{|Q_0|}(B)$ is an impression of A with B an integral domain and $\tau(e_i) = E_{ii}$ for each $i \in Q_0$, then the center of A is isomorphic to R and is depicted by S .

Proof. By [B, Lemma 2.4], the maximal k -dimension of the simple A -modules is $|Q_0|$. Thus there exists a path $p_{ji} \notin I$ between any two vertices i, j of Q . Since τ is injective,

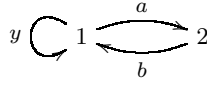


FIGURE 1.

$\bar{\tau}(p_{ji}) \neq 0$. We can therefore form a cycle $p_{1|Q_0|} \cdots p_{32}p_{21}$ which contains each vertex a subpath and whose τ -image is nonzero: indeed, since B is an integral domain we have

$$\bar{\tau}(p_{1|Q_0|} \cdots p_{32}p_{21}) = \bar{\tau}(p_{1|Q_0|}) \cdots \bar{\tau}(p_{32}) \bar{\tau}(p_{21}) \neq 0.$$

By the definition of impression, the morphism $\text{Max } B \rightarrow \text{Max } R$, $\mathfrak{q} \mapsto \mathfrak{q} \cap R$, is surjective. We may therefore apply Proposition 3.1. Finally, by [B, Lemma 2.1] $Z \cong R$. \square

Remark 3.4. The role of S is new: S is a commutative ring obtained from A that in most cases is not a central subring of A , but is closely related to the geometry of the center Z of A . When Z is noetherian then S is isomorphic to Z , while if Z is nonnoetherian then S properly contains Z .

Example 3.5. Consider the quiver algebra $A = kQ / \langle yba - bay \rangle$, with quiver given in figure 1. The labeling of arrows defines an impression $(\tau, B = k[a, b, z])$, and so we may apply Theorem 3.3. Letting $x := ab$, we find that the center of A is isomorphic to

$$R = k[\bar{\tau}(e_1 A e_1) \cap \bar{\tau}(e_2 A e_2)] = k + (x)S$$

and is depicted by

$$S = k[\bar{\tau}(e_1 A e_1) \cup \bar{\tau}(e_2 A e_2)] = k[x, y],$$

with R and S as in Examples 2.5, 2.7, and 2.12.⁴ We make two remarks:

- Even though A does not possess certain nice properties such as being a finitely-generated module over its center, the simple A -modules of maximal k -dimension (i.e., the simple modules with dimension vector $(1, 1)$ by [B, Lemma 5.1]) are nevertheless still parameterized by the smooth locus of $\text{Max } Z$, namely $(bc, a) \in k^* \times k$, which we naturally identify with $U \subset \text{Max } S$.
- The moduli space of θ -stable A -modules of dimension vector $(1, 1)$, for generic stability parameter θ , is precisely the ‘resolution’ $\text{Max } S$.

3.2. Depictions of non-cancellative toric superpotential algebras and the mesonic chiral ring. In this section we will consider superpotential algebras of non-cancellative brane tilings. Such algebras cannot admit impressions (τ, B) with B prime, but fortunately many of them (if not all) still have the property that their centers are isomorphic to $R = k\left[\bigcap_{i \in Q_0} \bar{\tau}(e_i A e_i)\right]$, where $\bar{\tau}$ is defined in (11) below.

⁴The path algebra kQ has been studied in [S, Proposition 6.1] in a different context.

Definition 3.6. A brane tiling is a quiver Q whose underlying graph \bar{Q} embeds into a two-dimensional real torus T^2 , such that each connected component of $T^2 \setminus \bar{Q}$ is simply connected and each cycle on the boundary of a connected component, called a *unit cycle*, is oriented and has length at least 2. A superpotential algebra A of a brane tiling Q is the quiver algebra kQ/I , where I is the (two-sided) ideal⁵

$$(10) \quad I := \langle d - d' \mid \exists a \in Q_1 \text{ such that } da \text{ and } d'a \text{ are unit cycles} \rangle \subset kQ.$$

Denote $p - q \in I$ by $p \equiv q$. A and Q are called *cancellative* if for all paths a, p, q with $h(a) = t(p) = t(q)$ (resp. $t(a) = h(p) = h(q)$), we have

$$p \equiv q \quad \text{whenever} \quad pa \equiv qa \quad (\text{resp. } ap \equiv aq).$$

Superpotential algebras of cancellative brane tilings are 3-Calabi-Yau algebras (e.g. [MR, Theorem 6.3], [D, Theorem 4.3]) and noncommutative crepant resolutions (e.g. [Bo, Theorem 10.2]). Moreover, their centers are 3-dimensional normal toric Gorenstein singularities ([Br], [B, section 4, for square brane tilings]). In contrast, superpotential algebras of non-cancellative brane tilings are not finitely-generated modules over their centers, and their centers are nonnoetherian. Cancellation in the context of brane tilings first appeared in [MR, Condition 4.12], and was expanded upon in [D, Definition 2.5, Lemma 7.3]. See [Bo, Theorem 11.1] for equivalent notions.

Remark 3.7. Any superpotential algebra of a brane tiling that admits an impression (τ, B) with B an integral domain will be cancellative. Indeed, if $pa \equiv qa$ then $\bar{\tau}(p)\bar{\tau}(a) = \bar{\tau}(pa) = \bar{\tau}(qa) = \bar{\tau}(q)\bar{\tau}(a)$, so $\bar{\tau}(p) = \bar{\tau}(q)$, so $\tau(p) = \bar{\tau}(p)E_{h(p), h(a)} = \bar{\tau}(q)E_{h(p), h(a)} = \tau(q)$, whence $p \equiv q$ by the injectivity of τ .

Throughout, denote by u_i a unit cycle at i , and by σ the $\bar{\tau}$ -image of a (any) unit cycle in Q . The following are well-known properties of the unit cycles; see for example [MR, Lemmas 4.2, 4.4, 4.5, 4.6].

Lemma 3.8. *Let $A = kQ/I$ be a superpotential algebra of any brane tiling (cancellative or not). Then*

- *The element $u := \sum_{i \in Q_0} u_i + I$ is in the center of A .*
- *If u_i, u'_i are two unit cycles at $i \in Q_0$ then $u_i - u'_i \in I$.*
- *If $p \in e_i kQ e_i$ is a cycle whose lift p^+ is a cycle in \tilde{Q} , then $p \equiv u_i^n$ for some $n \geq 0$.*

The following definition formalizes a notion of ‘Higgsing’ in quiver gauge theories, and we will use this notion to determine depictions of the centers of superpotential algebras of non-cancellative brane tilings.

⁵In [Bo, Definition 3.2], the unit cycles are required to have length at least 3. We will encounter examples in section 3.3 where it is necessary to allow cancellative algebras to have unit cycles of length 2. This poses no problem, however: if Q has unit cycles of length 2 and Q' is obtained from Q by deleting these two-cycles, then their corresponding superpotential algebras are equal (though their path algebras are not).

Definition 3.9. Given a brane tiling Q and a subset of arrows $Q_1^* \subseteq Q_1$, form the *contracted quiver* Q' by identifying the three paths a , $h(a)$, and $t(a)$, for each $a \in Q_1^*$. Denote by $\psi : kQ \rightarrow kQ'$ the k -homomorphism defined by sending a path in kQ to the corresponding path in kQ' .

If $d - d'$ is a minimal generator of I , that is, there exists an $a \in Q_1$ such that ad and ad' are unit cycles, then $\psi(d - d') = \psi(d) - \psi(d') \in I'$. Indeed, if $a \in Q_1^*$ then this follows from Lemma 3.8, and is trivial otherwise. Therefore $\psi(I) \subseteq I'$, and so ψ descends to a k -homomorphism $A = kQ/I \rightarrow A' = kQ'/I'$, which we also denote by ψ . It is clear that ψ is an algebra homomorphism on the corner rings $e_i A e_i$, $i \in Q_0$. We say ψ is a *contraction* on Q_1^* .

For the remainder of this section, let $A = kQ/I$ and $A' = kQ'/I'$ be superpotential algebras of brane tilings that are respectively non-cancellative and cancellative. Denote by Z and Z' the respective centers of A and A' . Suppose A' admits an impression $\tau : A' \hookrightarrow M_{|Q_0|}(B)$, where B is a polynomial ring, $\tau(e_i) = E_{ii}$ for each $i \in Q_0$, and $\bar{\tau}(a) \in B$ is a monomial for each $a \in Q_1$. Further, suppose $\psi : A \rightarrow A'$ is a contraction on a set of arrows $Q_1^* \subset Q_1$. We introduce the k -homomorphism $\tau : A \rightarrow M_{|Q_0|}(B)$ defined by

$$(11) \quad \bar{\tau}(a) := \bar{\tau}'(\psi(a)) \quad \text{for each } a \in e_j A e_i, \quad i, j \in Q_0.$$

Since τ' is an algebra homomorphism and ψ is an algebra homomorphism on each $e_i A e_i$, $i \in Q_0$, τ is also an algebra homomorphism on each $e_i A e_i$.

Let R, S and R', S' be defined by (8), (9) with τ and τ' respectively. Since A' is cancellative, it is well known that $Z' \cong R' = S'$ since there is an isomorphism of corner rings $e_i A e_i \cong e_j A e_j$ for each $i, j \in Q_0$. In Theorem 3.18 we will show that (i) $Z \cong R$ as algebras; (ii) $S = S'$; and (iii) R is depicted by S .

Physics Remark 3.10. In a 4-dimensional $\mathcal{N} = 1$ quiver gauge theory with quiver Q , the algebra generated by the cycles in Q modulo the F-flatness constraints, that is, the defining generators of I , is known as the ‘mesonic chiral ring’. In the context of these theories, the algebra S is similar to, if not a formalized definition of, the mesonic chiral ring.

Furthermore, the Higgsing considered here is presumably related to RG flow: we start with a non-superconformal (strongly coupled) quiver theory Q , give an arrow $\delta \in Q_1^*$ a nonzero vev, and end with a theory Q' that lies at a conformal fixed point. It is also possible to Higgs between two cancellative brane tilings, where quite often a \mathbb{P}^n in a partial resolution of $\text{Max } S$ is blown-down. In this case we expect that $S \neq S'$.

Let $\pi : \mathbb{R}^2 \rightarrow T^2$ be the canonical projection, and $\tilde{Q} := \pi^{-1}(Q) \subset \mathbb{R}^2$ the covering quiver (or ‘periodic quiver’) of Q . Fix a fundamental domain D of \tilde{Q} . For each path p in Q , denote by p^+ the unique path in \tilde{Q} with tail in D satisfying $\pi(p^+) = p$. For

a vertex j in the covering quiver \tilde{Q} , denote by \vec{j} the corresponding vector in \mathbb{R}^2 . We introduce the following definition.

Definition 3.11. We say ψ or Q_1^* is *adequate* if the following conditions hold:

- (1) If the lifts of two paths p and q contain no cyclic proper subpaths modulo I and bound a compact region whose interior does not contain the lift of any arrow in Q_1^* , then $p \equiv q$.
- (2) For each $i, j \in \tilde{Q}_0'$ satisfying $\pi(i) = \pi(j)$, there is a cycle $p \in kQ'$ whose $\bar{\tau}$ -image is not divisible by σ , does not contain the ψ -image of any arrow in Q_1^* , and satisfies

$$\vec{h}(p^+) - \vec{t}(p^+) = \vec{j} - \vec{i}.$$

Henceforth we will assume ψ is adequate unless stated otherwise.

Question 3.12. Do all non-cancellative brane tilings adequately contract to cancellative brane tilings whose algebras admit impressions?

Remark 3.13. We do not know of an example where condition 1 is not satisfied, though it is a non-trivial condition (see Proposition 3.20).

Lemma 3.14. *If $\psi : A \rightarrow A'$ is a contraction with A' cancellative and $c \in A$ is a cycle of positive length, then $\psi(c)$ is not a vertex.*

Proof. Suppose c is a cycle of positive length that contracts to a vertex. Then the lift c^+ must be a cycle: otherwise the underlying graph of $\psi(Q) = Q'$ could not embed into a two-torus since we are assuming Q' is non-degenerate. Furthermore, the unit cycle $u_{t(c)}$ must also contract to a vertex, for otherwise again the underlying graph of Q' could not embed into a two-torus.

Let $i \in Q_0$ be a vertex that is not the head or tail of any arrow in Q_1^* , and let p be a path from i to $t(c)$. Then by Lemma 3.8,

$$\psi(p)\psi(u_i) = \psi(pu_i) \equiv \psi(u_{t(c)}p) = \psi(u_{t(c)})\psi(p) = \psi(p).$$

Since Q' is cancellative, $\psi(u_i^n) = e_{\psi(i)}$. This contradicts our choice of i . \square

Lemma 3.15. *Let $p, q \in e_j k Q e_i$ be paths such that $\psi(p) \equiv \psi(q)$. Then their lifts p^+ and q^+ bound a compact region \mathcal{R} in the covering quiver $\tilde{Q} \subset \mathbb{R}^2$ of Q . Furthermore, if the interior of \mathcal{R} does not contain the lift of any arrow in Q_1^* , then $p \equiv q$.*

Proof. Suppose $\psi(p) \equiv \psi(q)$. Then their lifts $\psi(p)^+$ and $\psi(q)^+$ have coincident heads and tails. By Lemma 3.14, there is no cycle $c \in e_i k Q e_i$ of positive length satisfying $\psi(c) = e_{\psi(i)}$. Therefore p^+ and q^+ have coincident heads and tails as well, so p^+ and q^+ bound a compact region \mathcal{R} .

Suppose $p \not\equiv q$. Since $\psi(p) \equiv \psi(q)$ and the relations (10) are ‘homotopy relations’, the lift of some $\delta \in Q_1^*$ must lie in the interior of \mathcal{R} as an obstruction. \square

Proposition 3.16. *Let $\gamma \in B$ be a monomial. Suppose that for each $i \in Q_0$ there is a cycle $c_i \in e_i k Q e_i$ such that $\bar{\tau}(c_i) = \gamma$. Then $\sum_{i \in Q_0} c_i + I \in Z$. Furthermore, if $d \in e_i k Q e_i$ is another cycle satisfying $\bar{\tau}(d) = \gamma$ then $d \equiv c_i$.*

Proof. For the first claim, it suffices to show that for each arrow $a \in Q_1$ we have $ac_{t(a)} \equiv c_{h(a)}a$. Fix $a \in Q_1$.

First suppose $(ac_{t(a)})^+$ and $(c_{h(a)}a)^+$ do not contain cyclic proper subpaths. Then we may write $\mathcal{R} = \cup_j \mathcal{P}_j \subset \mathbb{R}^2$, where each \mathcal{P}_j is a closed region bounded by two paths p_j^+ and q_j^+ with no cyclic proper subpaths; the intersection of interiors is empty, $\mathcal{P}_j^\circ \cap \mathcal{P}_k^\circ = \emptyset$ for $j \neq k$; and the lift of any arrow in Q_1^* that lies in \mathcal{R} lies on the boundary of some \mathcal{P}_j . Therefore, by condition 1 in Definition 3.11, $p_j \equiv q_j$ for each j , whence $ac_{t(a)} \equiv c_{h(a)}a$.

Now suppose $(ac_{t(a)})^+$ or $(c_{h(a)}a)^+$ contains a cyclic proper subpath, say $ac_{t(a)} = p_2 q p_1$, where q^+ is a cycle. By Lemma 3.8, $q \equiv u_{h(p_1)}^m$ for some $m \geq 1$. Thus, since $\sum_{i \in Q_0} u_i + I$ is in the center of A , $ac_{t(a)} \equiv p_2 p_1 u_{t(a)}^m$. Similarly $c_{h(a)}a \equiv r u_{t(a)}^n$ for some path r and $n \geq 0$. Therefore the previous argument with $p_2 p_1$ and r in place of $ac_{t(a)}$ and $c_{h(a)}a$ implies that $p_2 p_1 \equiv r$. But $\bar{\tau}(p_2 p_1) \sigma^m = \bar{\tau}(ac_{t(a)}) = \bar{\tau}(c_{h(a)}a) = \bar{\tau}(r) \sigma^n$. Thus $m = n$ since B is a polynomial ring and $\bar{\tau}(r) = \bar{\tau}(p_2 p_1) \neq 0$. This yields $ac_{t(a)} \equiv c_{h(a)}a$.

The second claim follows from the same argument with c_i and d in place of $ac_{t(a)}$ and $c_{h(a)}a$. \square

Proposition 3.17. *The subalgebras S and S' of B are equal.*

Proof. Here we use condition 2 in Definition 3.11.

It is clear that $S \subseteq S'$. To show the converse, suppose that $\gamma \in S'$. Since A' is cancellative, S' is generated by monomials in B (S' is toric), and therefore we may suppose γ is a monomial. Furthermore, A' cancellative implies $S' = R'$. Therefore for each $i \in Q'_0$ there is a cycle $c_i \in e_i k Q' e_i$ such that $\bar{\tau}'(c_i) = \gamma$.

If $\gamma = \alpha\beta$ with $\alpha, \beta \in S'$, then either γ is in S , or α or β is not in S . Therefore we lose no generality in assuming γ is irreducible in S' . In particular $\sigma \nmid \gamma$ in S' since $\sigma \in S'$.

Fix $i \in Q_0$. By condition 2 there is a cycle $p \in k Q'$ with the following properties: (i) $\bar{\tau}(p)$ is not divisible by σ ; (ii) p does not contain any vertex $\psi(\delta) \in Q'_0$, $\delta \in Q_1^*$; and (iii) p^+ and $c_{t(p)}^+$ have coincident heads and tails in \tilde{Q}' .

(iii) implies that $\gamma = \bar{\tau}(p) \sigma^n$ for some $n \in \mathbb{Z}$ by Lemma 3.8. Moreover, (i) implies $n \geq 0$. But we are supposing $\sigma \nmid \gamma$ in S' as well, so $n = 0$. Furthermore, (ii) implies that p is the ψ -image of a path q in Q , whence $\bar{\tau}'(p) = \bar{\tau}(q) \in S$. Therefore $\gamma = \bar{\tau}'(p) \in S$. \square

Theorem 3.18. *Let $\psi : A \rightarrow A'$ be an adequate contraction between superpotential algebras of brane tilings, where A' is cancellative and A is not. Further suppose A' admits an impression (τ', B) , with B a polynomial ring and $\bar{\tau}'(a) \in B$ a monomial*

for each $a \in Q'_1$. Define the k -homomorphism $\tau : A \rightarrow M_{|Q_0|}(B)$ by (11), and let R, S and R', S' be as in (8), (9) with τ and τ' respectively. Then

$$Z \cong R \subsetneq S = S' = R' \cong Z'.$$

Furthermore, R is depicted by S . In particular, the ‘mesonic chiral ring’ of A , namely S , is a depiction of its center Z .

Proof. Denote by $\mathbf{1}$ the identity matrix in $M_{|Q_0|}(B)$. Recall that the k -homomorphism $\tau : A \rightarrow M_{|Q_0|}(B)$ is an algebra homomorphism on each $e_i A e_i$, $i \in Q_0$. We will show that the restriction

$$(12) \quad \tau : Z \rightarrow R\mathbf{1},$$

is an algebra isomorphism.

(i) The map (12) is well-defined, i.e., $\tau(Z) \subseteq R\mathbf{1}$:

Suppose $c \in Z$. Since c commutes with the vertex idempotents, c must be a sum of cycles: $c = \sum_{i \in Q_0} c_i$ with each $c_i \in e_i A e_i$. Let p be a path. Since τ is an algebra homomorphism on each $e_i A e_i$, we have

$$(13) \quad \bar{\tau}(p)\bar{\tau}(c_{t(p)}) = \bar{\tau}(pc_{t(p)}) = \bar{\tau}(pc) = \bar{\tau}(cp) = \bar{\tau}(c_{h(p)}p) = \bar{\tau}(c_{h(p)})\bar{\tau}(p) \in B.$$

Furthermore, $\bar{\tau}(p) = \bar{\tau}'(\psi(p))$ is nonzero since τ' is an impression of A' . Thus, since B is an integral domain, (13) implies $\bar{\tau}(c_{h(p)}) = \bar{\tau}(c_{t(p)})$. But there is a path between each pair of vertices in Q that is nonzero modulo I , and therefore $\tau(c) \in R\mathbf{1}$.

(ii) The map (12) is surjective, i.e., $R\mathbf{1} \subseteq \tau(Z)$:

In the following, by monomial, path, or cycle, we mean a scalar multiple thereof. We first show that R is generated by a set of monomials in B . Suppose $\sum_{j=1}^m \beta_j \in R$, with each β_j a monomial. By the definition of R , for each $i \in Q_0$ there exists a $b \in e_i k Q e_i$ such that $\bar{\tau}(b) = \sum_j \beta_j$. Suppose $b = \sum_{\ell=1}^n c_\ell$ for some cycles $c_\ell \in e_i k Q e_i$. By assumption, the $\bar{\tau}$ -image of any arrow is a monomial in B . Thus the $\bar{\tau}$ -image of any path is a monomial in B since B is a polynomial ring. Therefore, by k -linearity of $\bar{\tau}$,

$$\bar{\tau}(b) = \sum_{\ell} \bar{\tau}(c_\ell) = \sum_{\ell} \gamma_\ell,$$

where each $\gamma_\ell := \bar{\tau}(c_\ell)$ is a monomial since c_ℓ is a path. But then

$$\sum_{\ell=1}^n \gamma_\ell = \bar{\tau}(b) = \sum_{j=1}^m \beta_j.$$

Since B is a polynomial ring, $n = m$ and (possibly re-indexing) $\gamma_j = \beta_j$ for $1 \leq j \leq m$. β_j is therefore the $\bar{\tau}$ -image of a cycle in $e_i k Q e_i$. Since $i \in Q_0$ was arbitrary, $\beta_j \in R$, proving our claim.

Now let γ be a monomial in R . As we have just shown, γ is the $\bar{\tau}$ -image of an element $c = \sum_{i \in Q_0} c_i$ with each $c_i \in e_i k Q e_i$ a cycle whose $\bar{\tau}$ -image is γ . By

Proposition 3.16, $c + I \in Z$, so $\gamma \mathbf{1} \in \tau(Z)$. Since R is generated by a set of monomials in B , we have $R\mathbf{1} \subseteq \tau(Z)$.

(iii) *The map (12) is injective:*

If $b + I$ and $c + I$ are in Z and satisfy $\tau(b) = \tau(c)$ then $b \equiv c$ by Proposition 3.16.

Since (τ', B) is an impression of A' , $Z' \cong R'$ by [B, Lemma 2.1], and $R' = S'$ by [B, Theorem 2.11]. Furthermore, since (τ', B) is an impression of A' , the map $\text{Max } B \rightarrow \text{Max } Z' = \text{Max } S$, $\mathbf{q} \mapsto \mathbf{q} \cap Z'$, is surjective. Since $R \subseteq S \cong Z'$, the map $\text{Max } S \rightarrow \text{Max } R$, $\mathbf{n} \mapsto \mathbf{n} \cap R$, is surjective. Therefore S is a depiction of $Z \cong R$ by Proposition 3.1. \square

Corollary 3.19. *Suppose the hypotheses of Theorem 3.18 hold. Then the center Z of A is birational to the normal toric Gorenstein singularity $S = S'$ and contains a positive dimensional subvariety that is identified as a single (closed) point.*

The following proposition gives an infinite class of examples for which Theorem 3.18 is applicable. A notable example is the quiver for the cone over Q^{111} , given in Example 3.3.ii below, which has been studied in the context of Chern-Simons quiver gauge theories in string theory.

Proposition 3.20. *Suppose Q contains no loops, and for each $\delta \in Q_1^*$, $t(\delta)$ or $h(\delta)$ has indegree and outdegree 1. Then condition 1 in Definition 3.11 is satisfied.*

Proof. Condition 1 is only non-trivial in the following case: Suppose p and $q = r_2 s r_1$ are paths whose lifts contain no cyclic proper subpaths modulo I , bound a compact region whose interior does not contain the lift of any arrow in Q_1^* , and such that $s\delta$ is a unit cycle for some $\delta \in Q_1^*$. For condition 1 to be satisfied we must show that $p \equiv q$, which is non-trivial since it implies that σ divides the $\bar{\tau}$ -image of *every* such path in Q .

Suppose $t(\delta)$ has indegree and outdegree 1. If $t(\delta) \neq h(q)$, i.e., $r_2 \neq e_{h(q)}$, then q must contain the unit cycle δs since $h(s) = t(\delta)$ has outdegree 1. Moreover, since p^+ and q^+ have coincident heads and tails (they bound a compact region) and have no cyclic proper subpaths, and A' is cancellative, $\psi(p) \equiv \psi(q)$. Therefore $p \equiv q$ by Lemma 3.15.

Otherwise $h(p) = h(q) = t(\delta)$ has indegree 1, so p and q have the same leftmost arrow subpath a , say $p = ap'$ and $q = aq'$. The paths p' and q' then bound a compact region whose interior does not contain the lift of any arrow in Q_1^* , and δ only meets q' at its head since a is not a loop. Therefore $p' \equiv q'$, whence $p \equiv q$.

The case where $h(\delta)$ has indegree and outdegree 1 is similar. \square

3.3. Examples. We now consider some ‘nonnoetherian deformations’ of square superpotential algebras. A superpotential algebra $A = kQ/I$ of a brane tiling is *square* if the underlying graph of Q is a square grid graph with vertex set $\mathbb{Z} \times \mathbb{Z}$, and with at most one diagonal edge in each unit square.

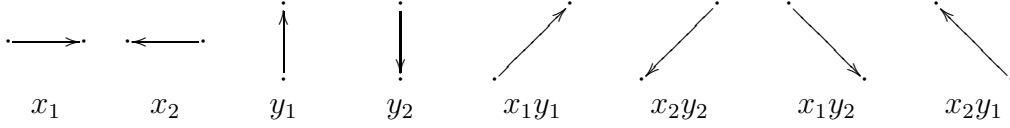


FIGURE 2. A labeling of arrows in the quiver of a square superpotential algebra that specifies an impression.

Any square superpotential algebra A admits an impression $(\tau, B = k[x_1, x_2, y_1, y_2])$, where for each arrow a in the covering quiver \tilde{Q} , $\bar{\tau}(a)$ is the monomial corresponding to the orientation of a given in figure 2 [B, Theorem 3.7]. Specifically, $\tau : A \rightarrow M_{|Q_0|}(B)$ is the k -algebra homomorphism defined on the generating set $Q_0 \cup Q_1$ by

$$e_i \mapsto E_{ii} \text{ for each } i \in Q_0 \text{ and } a \mapsto \bar{\tau}(a)E_{h(a),t(a)} \text{ for each } a \in Q_1.$$

If Q only possesses three arrow orientations, say up, left, and right-down, then we may label the respective arrows by x , y , and z , and obtain an impression $(\tau, k[x, y, z])$. In either case, A is cancellative by Remark 3.7.

Consider the four examples of nonnoetherian deformations of square superpotential algebras given in figure 3. In each example, the quiver labeled (a) is non-cancellative and contracts to the cancellative quiver Q' labeled (b). In examples (iii) and (iv), quiver (c) is obtained from (b) by removing all 2-cycles. The superpotential algebras corresponding to (b) and (c) are equal, although their path algebras (without relations) are not. The non-cancellative quivers (a) first appeared in [DHP, Table 5, 2.3]; [FHPR, Section 4], [FKR]; [Bo, Example 3.2]; and [DHP, Table 6, 2.6], respectively, each in a different form from what is shown here.

In example (i), the impression ring B equals $k[x, y, z]$ since there are only three orientations of arrows in Q' , while in examples (ii)-(iv), B equals $k[x_1, x_2, y_1, y_2]$. It is easy to check that in each example, the contraction ψ is adequate. Therefore the following hold:

$$\begin{aligned} \text{(i):} \quad & Z' \cong S = k[x^2, y^2, xy, z] = R' = S' \\ & Z \cong R = k + (x^2, y^2, xy)S \\ \text{(ii):} \quad & Z' \cong S = k[x_1y_1, x_1y_2, x_2y_1, x_2y_2] = R' = S' \\ & Z \cong R = k + (x_1^2y_1y_2, x_1x_2y_1y_2, x_2^2y_1y_2)S \\ \text{(iii):} \quad & Z' \cong S = k[x_1y_1, x_2y_1, x_1y_2, x_2y_2] = R' = S' \\ & Z \cong R = k + (x_1y_1, x_2y_1)S \\ \text{(iv):} \quad & Z' \cong S = k[x_1y_1, x_2y_2, x_1^2y_2^2, x_2^2y_1^2] = R' = S' \\ & Z \cong R = k + (x_2y_2, x_1^2y_2^2, x_2^2y_1^2)S \end{aligned}$$

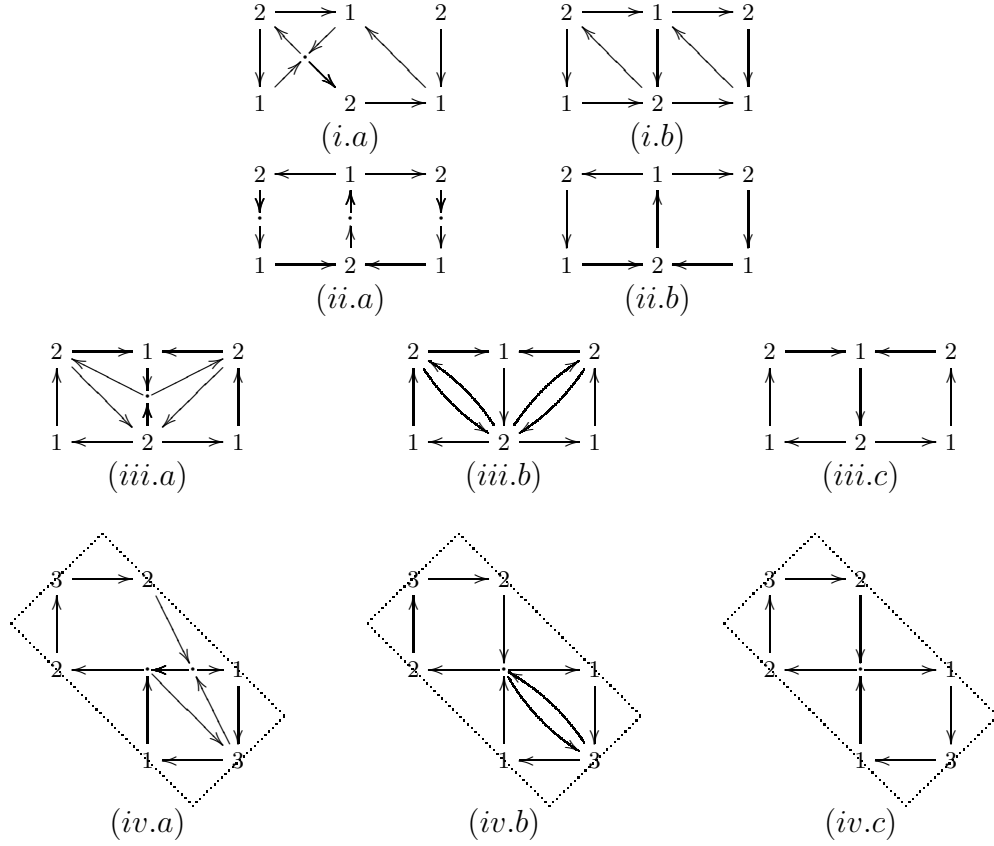


FIGURE 3. Nonnoetherian deformations of some square superpotential algebras. Each quiver is drawn on a torus, and the contracted arrows are drawn in bold.

By Theorem 3.18, in each example $R \cong Z$ is depicted by $S = S' = R' \cong Z'$. Moreover, in example (i) R is depicted by the abelian orbifold \mathbb{C}^3/μ_2 of type $\frac{1}{2}(1, 1, 0)$, while in examples (ii) and (iii) R is depicted by the conifold (quadric cone).

In example (iii), if both of the upward pointing arrows are contracted, then the resulting quiver Q' consists of one vertex and 3 loops—the standard \mathbb{C}^3 brane tiling—and is therefore cancellative. However, for this choice of Q'_1 , condition 2 in Definition 3.11 is not satisfied, and the conclusions of Theorem 3.18 do not hold since

$$S = k[x, y, xz, yz] \neq k[x, y, z] = S'.$$

In a forthcoming paper [B2], some questions regarding the representation theory of non-cancellative brane tilings will be addressed.

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